This examination paper consists of THREE sections: Module 1, Module 2 and Module 3.

Each section consists of 2 questions.
The maximum mark for each Module is 50.
The maximum mark for this examination is 150.
This examination consists of 6 printed pages.

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. DO NOT open this examination paper until instructed to do so.

2. Answer ALL questions from the THREE sections.

3. Write your solutions, with full working, in the answer booklet provided.

4. Unless otherwise stated in the question, any numerical answer that is not exact MUST be written correct to three significant figures.

Examination Materials Permitted

Graph paper (provided)
Mathematical formulae and tables (provided) – Revised 2012
Mathematical instruments
Silent, non-programmable, electronic calculator
SECTION A (Module 1)

Answer BOTH questions.

1. (a) The expression \( f(x) = 2x^3 - px^2 + qx - 10 \) is divisible by \( x - 1 \) and has a remainder \(-6\) when divided by \( x + 1 \).

Find

(i) the values of the constants \( p \) and \( q \) [7 marks]
(ii) the factors of \( f(x) \). [3 marks]

(b) Find positive integers \( x \) and \( y \) such that

\[ (\sqrt{x} + \sqrt{y})^2 = 16 + \sqrt{240}. \] [8 marks]

(c) (i) Solve, for real values of \( x \), the inequality

\[ |3x - 7| \leq 5. \] [5 marks]

(ii) Show that no real solution, \( x \), exists for the inequality \( |3x - 7| + 5 \leq 0 \). [2 marks]

Total 25 marks

2. (a) The function \( f \) on \( \mathbb{R} \) is defined by

\[ f: x \rightarrow x^2 - 3. \]

(i) Find, in terms of \( x \), \( f(f(x)) \). [3 marks]

(ii) Determine the values of \( x \) for which

\[ f(f(x)) = f(x + 3). \] [6 marks]

(b) The roots of the equation \( 4x^2 - 3x + 1 = 0 \) are \( \alpha \) and \( \beta \).

Without solving the equation

(i) write down the values of \( \alpha + \beta \) and \( \alpha\beta \) [2 marks]

(ii) find the value of \( \alpha^2 + \beta^2 \) [2 marks]

(iii) obtain a quadratic equation whose roots are \( \frac{2}{\alpha^2} \) and \( \frac{2}{\beta^2} \). [5 marks]
(c) Without the use of calculators or tables, evaluate

(i) \[ \log_{10} \left( \frac{1}{3} \right) + \log_{10} \left( \frac{3}{5} \right) + \log_{10} \left( \frac{5}{7} \right) + \log_{10} \left( \frac{7}{9} \right) + \log_{10} \left( \frac{9}{10} \right) \] [3 marks]

(ii) \[ \sum_{r=1}^{99} \log_{10} \left( \frac{r}{r+1} \right). \] [4 marks]

Total 25 marks

SECTION B (Module 2)

Answer BOTH questions.

3. (a) (i) Given that \( \cos (A + B) = \cos A \cos B - \sin A \sin B \) and \( \cos 2\theta = 2 \cos^2 \theta - 1 \), prove that

\[ \cos 3\theta = 2 \cos \theta \left[ \cos^2 \theta - \sin^2 \theta - \frac{1}{2} \right]. \] [7 marks]

(ii) Using the appropriate formula, show that

\[ \frac{1}{2} [\sin 6\theta - \sin 2\theta] \equiv (2 \cos^2 2\theta - 1) \sin 2\theta. \] [5 marks]

(iii) Hence, or otherwise, solve \( \sin 6\theta - \sin 2\theta = 0 \) for \( 0 \leq \theta \leq \frac{\pi}{2} \). [5 marks]

(b) Find ALL possible values of \( \cos \theta \) such that \( 2 \cot^2 \theta + \cos \theta = 0. \) [8 marks]

Total 25 marks
4. (a) (i) Determine the Cartesian equation of the curve, C, defined by the parametric equations \( y = 3 \sec \theta \) and \( x = 3 \tan \theta \). \[5 \text{ marks}\]

(ii) Find the points of intersection of the curve \( y = \sqrt{10x} \) with C. \[9 \text{ marks}\]

(b) Let \( \mathbf{p} \) and \( \mathbf{q} \) be two position vectors with endpoints \((-3, 4)\) and \((-1, 6)\) respectively.

(i) Express \( \mathbf{p} \) and \( \mathbf{q} \) in the form \( xi + yj \). \[2 \text{ marks}\]

(ii) Obtain the vector \( \mathbf{p} - \mathbf{q} \). \[2 \text{ marks}\]

(iii) Calculate \( \mathbf{p} \cdot \mathbf{q} \). \[2 \text{ marks}\]

(iv) Let the angle between \( \mathbf{p} \) and \( \mathbf{q} \) be \( \theta \). Use the result of (iii) above to calculate \( \theta \) in degrees. \[5 \text{ marks}\]

Total 25 marks
SECTION C (Module 3)

Answer BOTH questions.

5. (a) (i) Find the values of x for which \( \frac{x^3 + 8}{x^2 - 4} \) is discontinuous. [2 marks]

   (ii) Hence, or otherwise, find
   \[
   \lim_{x \to -2} \frac{x^3 + 8}{x^2 - 4}.
   \]
   [3 marks]

   (iii) By using the fact that \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \), or otherwise, find,
   \[
   \lim_{x \to 0} \frac{2x^3 + 4x}{\sin 2x}.
   \]
   [5 marks]

(b) The function \( f \) on \( \mathbb{R} \) is defined by
   \[
   f(x) = \begin{cases} 
   x^2 + 1, & x > 1, \\
   4 + px, & x < 1.
   \end{cases}
   \]

   (i) Find

   a) \( \lim_{x \to 1^+} f(x) \) [2 marks]

   b) the value of the constant \( p \) such that \( \lim_{x \to 1} f(x) \) exists. [4 marks]

   (ii) Hence, determine the value of \( f(1) \) for \( f \) to be continuous at the point \( x = 1 \). [1 mark]

(c) A chemical process in a manufacturing plant is controlled by the function
   \[
   M = ut^2 + \frac{v}{t^2}
   \]
   where \( u \) and \( v \) are constants.

   Given that \( M = -1 \) when \( t = 1 \) and that the rate of change of \( M \) with respect to \( t \) is \( \frac{35}{4} \) when \( t = 2 \), find the values of \( u \) and \( v \). [8 marks]

Total 25 marks
6. (a) (i) Given that \( y = \sqrt{4x^2 - 7} \), show that \( y \frac{dy}{dx} = 4x \). [3 marks]

(ii) Hence, or otherwise, show that
\[
y \frac{d^2y}{dx^2} + \left( \frac{dy}{dx} \right)^2 = 4.
\] [3 marks]

(b) The curve, \( C \), passes through the point \((-1, 0)\) and its gradient at the point \((x, y)\) is given by
\[
\frac{dy}{dx} = 3x^2 - 6x.
\]

(i) Find the equation of \( C \). [4 marks]

(ii) Find the coordinates of the stationary points of \( C \). [3 marks]

(iii) Determine the nature of EACH stationary point. [3 marks]

(iv) Find the coordinates of the points \( P \) and \( Q \) at which the curve \( C \) meets the \( x \)-axis. [5 marks]

(v) Hence, sketch the curve \( C \), showing
a) the stationary points
b) the points \( P \) and \( Q \). [4 marks]

Total 25 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.

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