This examination paper consists of THREE sections: Module 1, Module 2 and Module 3.

Each section consists of 2 questions.
The maximum mark for each Module is 50.
The maximum mark for this examination is 150.
This examination consists of 6 printed pages.

**INSTRUCTIONS TO CANDIDATES**

1. **DO NOT** open this examination paper until instructed to do so.
2. Answer **ALL** questions from the **THREE** sections.
3. Write your solutions, with full working, in the answer booklet provided.
4. Unless otherwise stated in the question, any numerical answer that is not exact **MUST** be written correct to three significant figures.

**Examination Materials Permitted**

Graph paper (provided)
Mathematical formulae and tables (provided) – **Revised 2008**
Mathematical instruments
Silent, non-programmable, electronic calculator
SECTION A (Module 1)

Answer BOTH questions.

1. (a) (i) Determine the values of the real number \( h \) for which the roots of the quadratic equation \( 4x^2 - 2hx + (8 - h) = 0 \) are real. [8 marks]

(ii) The roots of the cubic equation

\[ x^3 - 15x^2 + px - 105 = 0 \]

are \( 5 - k \), \( 5 \) and \( 5 + k \).

Find the values of the constants \( p \) and \( k \). [7 marks]

(b) (i) Copy the table below and complete by inserting the values for the functions \( f(x) = |x + 2| \) and \( g(x) = 2|x - 1| \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>(1)</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>(8)</td>
</tr>
<tr>
<td>(-2)</td>
<td>(1)</td>
</tr>
<tr>
<td>(-1)</td>
<td>(3)</td>
</tr>
<tr>
<td>(0)</td>
<td>(6)</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
</tr>
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<td>(2)</td>
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<tr>
<td>(3)</td>
<td>(2)</td>
</tr>
<tr>
<td>(4)</td>
<td>(2)</td>
</tr>
<tr>
<td>(5)</td>
<td>(2)</td>
</tr>
</tbody>
</table>

[4 marks]

(ii) Using a scale of 1 cm to 1 unit on both axes, draw on the same graph \( f(x) \) and \( g(x) \) for \(-3 \leq x \leq 5\). [4 marks]

(iii) Using the graphs, find the values of \( x \) for which \( f(x) = g(x) \). [2 marks]

Total 25 marks
2. (a) Without using calculators or tables, evaluate

\[ \sqrt{\frac{27^{10} + 9^{10}}{27^4 + 9^{11}}} \]  

[8 marks]

(b) (i) Prove that \( \log_m n = \frac{\log_{10} m}{\log_{10} n} \), for \( m, n \in \mathbb{N} \).  

[4 marks]

(ii) Hence, given that \( y = (\log_2 3) (\log_3 4) (\log_4 5) \ldots (\log_{31} 32) \), calculate the exact value of \( y \).  

[6 marks]

(c) Prove, by the principle of mathematical induction, that

\[ f(n) = 7^n - 1 \]

is divisible by 6, for all \( n \in \mathbb{N} \).  

[7 marks]

Total 25 marks

SECTION B (Module 2)

Answer BOTH questions.

3. (a) Let \( p = i - j \). If \( q = \lambda i + 2j \), find values of \( \lambda \) such that

(i) \( q \) is parallel to \( p \)  

[1 mark]

(ii) \( q \) is perpendicular to \( p \)  

[2 marks]

(iii) the angle between \( p \) and \( q \) is \( \frac{\pi}{3} \).  

[5 marks]

(b) Show that \[ \frac{1 - \cos 2A + \sin 2A}{1 + \cos 2A + \sin 2A} = \tan A. \]  

[6 marks]

(c) (i) Using the formula for \( \sin A + \sin B \), show that if \( t = 2 \cos \theta \) then \( \sin (n + 1) \theta = t \sin n\theta - \sin (n - 1) \theta \)  

[2 marks]

(ii) Hence, show that \( \sin 3\theta = (t^2 - 1) \sin \theta. \)  

[2 marks]

(iii) Using (c) (ii) above, or otherwise, find ALL solutions of \( \sin 3\theta = \sin \theta, 0 \leq \theta \leq \pi. \)  

[7 marks]

Total 25 marks

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22134020/CAPE 2008
4. (a) (i) The line $x - 2y + 4 = 0$ cuts the circle, $x^2 + y^2 - 2x - 20y + 51 = 0$ with centre $P$, at the points $A$ and $B$.

Find the coordinates of $P$, $A$ and $B$. \[6\text{ marks}\]

(ii) The equation of any circle through $A$ and $B$ is of the form

$$x^2 + y^2 - 2x - 20y + 51 + \lambda(x - 2y + 4) = 0$$

where $\lambda$ is a parameter.

A new circle $C$ with centre $Q$ passes through $P$, $A$ and $B$.

Find

a) the value of $\lambda$ \[2\text{ marks}\]

b) the equation of circle $C$ \[2\text{ marks}\]

c) the distance, $|PQ|$, between the centres \[3\text{ marks}\]

d) the distance $|PM|$ if $PQ$ cuts $AB$ at $M$. \[4\text{ marks}\]

(b) A curve is given by the parametric equations $x = 2 + 3 \sin t$, $y = 3 + 4 \cos t$.

Show that

(i) the Cartesian equation of the curve is

$$\frac{(x - 2)^2}{9} + \frac{(y - 3)^2}{16} = 1$$

\[3\text{ marks}\]

(ii) every point on the curve lies within or on the circle

$$(x - 2)^2 + (y - 3)^2 = 25.$$  \[5\text{ marks}\]

Total 25 marks
SECTION C (Module 3)

Answer BOTH questions.

5.  (a) Use L’Hopital’s rule to obtain \( \lim_{x \to 0} \frac{\sin 4x}{\sin 5x} \). [3 marks]

(b) (i) Given that \( y = \frac{x}{1 - 4x} \),

\[
\begin{align*}
\text{a) } & \quad \text{find } \frac{dy}{dx} \quad \text{[4 marks]} \\
\text{b) } & \quad \text{show that } x^2 \frac{dy}{dx} = y^2. \quad \text{[2 marks]}
\end{align*}
\]

(ii) Hence, or otherwise, show that \( x^2 \frac{d^2y}{dx^2} + 2 (x - y) \frac{dy}{dx} = 0 \). [3 marks]

(c) A rectangular box without a lid is made from thin cardboard. The sides of the base are 2x cm and 3x cm, and its height is h cm. The total surface area of the box is 200 cm².

(i) Show that \( h = \frac{20}{x} \) \(-\frac{3x}{5} \). [4 marks]

(ii) Find the height of the box for which its volume \( V \) cm³ is a maximum. [9 marks]

Total 25 marks
6. (a) Use the substitution $u = 3x^2 + 1$ to find $\int \frac{x \, dx}{\sqrt{3x^2 + 1}}$. [6 marks]

(b) A curve $C$ passes through the point $(3, -1)$ and has gradient $x^2 - 4x + 3$ at the point $(x, y)$ on $C$.

Find the equation of $C$. [4 marks]

(c) The figure below (not drawn to scale) shows part of the line $y + 2x = 5$ and part of the curve $y = x(4 - x)$ which meet at $A$. The line meets $Oy$ at $B$ and the curve cuts $Ox$ at $C$.

(i) Find the coordinates of $A$, $B$ and $C$. [6 marks]

(ii) Hence find the exact value of the area of the shaded region. [9 marks]

Total 25 marks

END OF TEST