This examination paper consists of THREE sections: Module 1, Module 2 and Module 3.

Each section consists of 2 questions.
The maximum mark for each Module is 50.
The maximum mark for this examination is 150.
This examination consists of 6 printed pages.

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. DO NOT open this examination paper until instructed to do so.
2. Answer ALL questions from the THREE sections.
3. Write your solutions, with full working, in the answer booklet provided.
4. Unless otherwise stated in the question, any numerical answer that is not exact MUST be written correct to three significant figures.

Examination Materials Permitted

Graph paper (provided)
Mathematical formulae and tables (provided) – Revised 2012
Mathematical instruments
Silent, non-programmable, electronic calculator

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.
SECTION A (Module 1)

Answer BOTH questions.

1. (a) Let $p$ and $q$ be two propositions. Construct a truth table for the statements

(i) $p \rightarrow q$ [1 mark]

(ii) $\neg (p \land q)$. [2 marks]

(b) A binary operator $\oplus$ is defined on a set of positive real numbers by

$$y \oplus x = y^2 + x^2 + 2y + x - 5xy.$$  

Solve the equation $2 \oplus x = 0$. [5 marks]

(c) Use mathematical induction to prove that $5^n + 3$ is divisible by 2 for all values of $n \in \mathbb{N}$. [8 marks]

(d) Let $f(x) = x^3 - 9x^2 + px + 16$.

(i) Given that $(x + 1)$ is a factor of $f(x)$, show that $p = 6$. [2 marks]

(ii) Factorise $f(x)$ completely. [4 marks]

(iii) Hence, or otherwise, solve $f(x) = 0$. [3 marks]

Total 25 marks
2. (a) Let \( A = \{x : x \in \mathbb{R}, x \geq 1\} \).

A function \( f : A \rightarrow \mathbb{R} \) is defined as \( f(x) = x^2 - x \). Show that \( f \) is one to one. [7 marks]

(b) Let \( f(x) = 3x + 2 \) and \( g(x) = e^{2x} \).

(i) Find

a) \( f^{-1}(x) \) and \( g^{-1}(x) \) [4 marks]

b) \( f(g(x)) \) (or \( f \circ g(x) \)). [1 mark]

(ii) Show that \( (f \circ g)^{-1}(x) = g^{-1}(x) \circ f^{-1}(x) \). [5 marks]

(c) Solve the following:

(i) \( 3x^2 + 4x + 1 \leq 5 \) [4 marks]

(ii) \( |x + 2| = 3x + 5 \) [4 marks]

Total 25 marks

SECTION B (Module 2)

Answer BOTH questions.

3. (a) (i) Show that \( \sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} \). [4 marks]

(ii) Hence, or otherwise, solve \( \sin 2\theta - \tan \theta = 0 \) for \( 0 \leq \theta \leq 2\pi \). [8 marks]

(b) (i) Express \( f(\theta) = 3 \cos \theta - 4 \sin \theta \) in the form \( r \cos (\theta + \alpha) \) where \( r > 0 \) and \( 0^\circ \leq \alpha \leq \frac{\pi}{2} \). [4 marks]

(ii) Hence, find

a) the maximum value of \( f(\theta) \) [2 marks]

b) the minimum value of \( \frac{1}{8 + f(\theta)} \). [2 marks]

(iii) Given that the sum of the angles \( A \), \( B \) and \( C \) of a triangle is \( \pi \) radians, show that

a) \( \sin A = \sin (B + C) \) [3 marks]

b) \( \sin A + \sin B + \sin C = \sin (A + B) + \sin (B + C) + \sin (A + C) \). [2 marks]

Total 25 marks

GO ON TO THE NEXT PAGE
4.  (a) A circle \( C \) is defined by the equation \( x^2 + y^2 - 6x - 4y + 4 = 0 \).

(i) Show that the centre and the radius of the circle, \( C \), are \((3, 2)\) and 3, respectively.  
[3 marks]

(ii)  
(a) Find the equation of the normal to the circle \( C \) at the point \((6, 2)\).  
[3 marks]

(b) Show that the tangent to the circle at the point \((6, 2)\) is parallel to the \(y\)-axis.  
[3 marks]

(b) Show that the Cartesian equation of the curve that has the parametric equations

\[ x = t^2 + t, \ y = 2t - 4 \]

is \(4x = y^2 + 10y + 24\).  
[4 marks]

(c) The points \( A (3, -1, 2), B (1, 2, -4) \) and \( C (-1, 1, -2) \) are three vertices of a parallelogram \( ABCD \).

(i) Express the vectors \( \vec{AB} \) and \( \vec{BC} \) in the form \( xi + yj + zk \).  
[3 marks]

(ii) Show that the vector \( \vec{r} = -16j - 8k \) is perpendicular to the plane through \( A, B \) and \( C \).  
[5 marks]

(iii) Hence, find the Cartesian equation of the plane through \( A, B \) and \( C \).  
[4 marks]

Total 25 marks
5. (a) A function \( f(x) \) is defined as \( f(x) = \begin{cases} x + 2, & x < 2 \\ x^2, & x > 2 \end{cases} \).

(i) Find \( \lim_{x \to 2} f(x) \). \[4 \text{ marks}\]

(ii) Determine whether \( f(x) \) is continuous at \( x = 2 \). Give a reason for your answer. \[2 \text{ marks}\]

(b) Let \( y = \frac{x^2 + 2x + 3}{(x^2 + 2)^3} \). Show that \( \frac{dy}{dx} = \frac{-4x^2 - 10x^2 - 14x + 4}{(x^2 + 2)^4} \). \[5 \text{ marks}\]

(c) The equation of an ellipse is given by \( x = 1 - 3 \cos \theta, \; y = 2 \sin \theta, \; 0 \leq \theta \leq 2\pi \).

Find \( \frac{dy}{dx} \) in terms of \( \theta \). \[5 \text{ marks}\]

(d) The diagram below (not drawn to scale) shows the curve \( y = x^2 + 3 \) and the line \( y = 4x \).

(i) Determine the coordinates of the points \( P \) and \( Q \) at which the curve and the line intersect. \[4 \text{ marks}\]

(ii) Calculate the area of the shaded region. \[5 \text{ marks}\]

Total 25 marks
6. (a) (i) By using the substitution $u = 1 - x$, find $\int x (1 - x)^3 \, dx$. [5 marks]

(ii) Given that $f(t) = 2 \cos t$, $g(t) = 4 \sin 5t + 3 \cos t$,

show that $\int [f(t) + g(t)] \, dt = \int f(t) \, dt + \int g(t) \, dt$. [4 marks]

(b) A sports association is planning to construct a running track in the shape of a rectangle surmounted by a semicircle, as shown in the diagram below. The letter $x$ represents the length of the rectangular section and $r$ represents the radius of the semicircle.

The perimeter of the track must be 600 metres.

(i) Show that $r = \frac{600 - 2x}{2 + \pi}$. [2 marks]

(ii) Hence, determine the length, $x$, that maximises the area enclosed by the track. [6 marks]

(c) (i) Let $y = -x \sin x - 2 \cos x + Ax + B$, where $A$ and $B$ are constants.

Show that $y'' = x \sin x$. [4 marks]

(ii) Hence, determine the specific solution of the differential equation

$y'' = x \sin x$,

given that when $x = 0$, $y = 1$ and when $x = \pi$, $y = 6$. [4 marks]

Total 25 marks

END OF TEST

FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.