CARIBBEAN EXAMINATIONS COUNCIL
ADVANCED PROFICIENCY EXAMINATION
PURE MATHEMATICS
UNIT 1 – PAPER 02

2 hours

23 MAY 2007 (p.m.)

This examination paper consists of THREE sections: Module 1, Module 2 and Module 3.

Each section consists of 2 questions.
The maximum mark for each section is 40.
The maximum mark for this examination is 120.
This examination consists of 6 pages.

INSTRUCTIONS TO CANDIDATES

1. **DO NOT** open this examination paper until instructed to do so.
2. Answer **ALL** questions from the **THREE** sections.
3. Unless otherwise stated in the question, any numerical answer that is not exact **MUST** be written correct to three significant figures.

Examination Materials

Mathematical formulae and tables
Electronic calculator
Graph paper

Copyright © 2006 Caribbean Examinations Council®
All rights reserved.

02134020/CAPE 2007
Section A (Module 1)

Answer BOTH questions.

1. (a) Let \( g(x) = x^4 - 9, \ x \in \mathbb{R} \). Find

(i) all the real factors of \( g(x) \) \hspace{1cm} \boxed{3 \text{ marks}}

(ii) all the real roots of \( g(x) = 0 \). \hspace{1cm} \boxed{1 \text{ mark}}

(b) The function \( f \) is defined by \( f(x) = x^4 - 9x^3 + 28x^2 - 36x + 16, \ x \in \mathbb{R} \); and \( u = x + \frac{4}{x}, \ x \neq 0 \).

(i) Express \( u^2 \) in terms of \( x \). \hspace{1cm} \boxed{3 \text{ marks}}

(ii) By writing \( f(x) = x^4 - 9x^3 + 28 - \frac{36}{x} + \frac{16}{x^2} \) and using the result from (b) (i) above, show that if \( f(x) = 0 \), then \( u^2 - 9u + 20 = 0 \). \hspace{1cm} \boxed{6 \text{ marks}}

(iii) Hence, determine the values of \( x \in \mathbb{R} \) for which \( f(x) = 0 \). \hspace{1cm} \boxed{7 \text{ marks}}

Total 20 marks

2. (a) Let \( S_n = \sum_{r=1}^{n} r \) for \( n \in \mathbb{N} \). Find the value of \( n \) for which \( 3S_{2n} = 11 S_n \). \hspace{1cm} \boxed{4 \text{ marks}}

\[
\left[ \text{Note: } \sum_{r=1}^{n} r = \frac{1}{2} n(n+1) \right]
\]

(b) The quadratic equation \( x^2 - px + 24 = 0, \ p \in \mathbb{R} \), has roots \( \alpha \) and \( \beta \), and the quadratic equation \( x^2 - 8x + q = 0, \ q \in \mathbb{R} \), has roots \( 2\alpha + \beta \) and \( 2\alpha - \beta \).

(i) Express \( p \) and \( q \) in terms of \( \alpha \) and \( \beta \). \hspace{1cm} \boxed{2 \text{ marks}}

(ii) Find the values of \( \alpha \) and \( \beta \). \hspace{1cm} \boxed{4 \text{ marks}}

(iii) Hence, determine the values of \( p \) and \( q \). \hspace{1cm} \boxed{2 \text{ marks}}

(c) Prove, by Mathematical Induction, that \( n^2 > 2n \) for all integers \( n \geq 3 \). \hspace{1cm} \boxed{8 \text{ marks}}

Total 20 marks

GO ON TO THE NEXT PAGE
Section B (Module 2)

Answer BOTH questions.

3. The circle shown in the diagram below (not drawn to scale) has centre C at (5, -4) and touches the y-axis at the point D. The circle cuts the x-axis at points A and B. The tangent at B cuts the y-axis at the point P.

(a) Determine

(i) the length of the radius of the circle [2 marks]
(ii) the equation of the circle [1 mark]
(iii) the coordinates of the points A and B, at which the circle cuts the x-axis [6 marks]
(iv) the equation of the tangent at B [4 marks]
(v) the coordinates of P. [2 marks]

(b) Show by calculation that \( PD = PB \). [5 marks]

Total 20 marks
4. (a) (i) Prove that \( \cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \). [4 marks]

(ii) Hence, show, without using calculators, that \( \tan 67\frac{1}{2}^\circ = 1 + \sqrt{2} \). [7 marks]

(b) In the triangle shown below, (not drawn to scale), \( \sin q = \frac{3}{5} \) and \( \cos p = \frac{5}{13} \).

Determine the exact values of

(i) \( \cos q \) [1 mark]

(ii) \( \sin p \) [1 mark]

(iii) \( \sin r \) [3 marks]

(iv) \( \cos (p + r) \). [4 marks]

Total 20 marks
Section C (Module 3)

Answer BOTH questions.

5. (a) Given that \( y = \sqrt{5x^2 + 3} \),

(i) obtain \( \frac{dy}{dx} \) [4 marks]

(ii) show that \( y \frac{dy}{dx} = 5x \) [2 marks]

(iii) hence, or otherwise, show that \( y \frac{d^2y}{dx^2} + \left( \frac{dy}{dx} \right)^2 = 5 \). [4 marks]

(b) At a certain port, high tides and low tides occur daily. Suppose \( t \) minutes after high tide, the height, \( h \) metres, of the tide above a fixed point is given by

\[
h = 2 \left( 1 + \cos \frac{\pi t}{450} \right), \quad 0 \leq t.
\]

[Note: High tide occurs when \( h \) has its maximum value and low tide when \( h \) has its minimum value.]

Determine

(i) the height of the tide when high tide occurs for the first time [2 marks]

(ii) the length of time which elapses between the first high tide and the first low tide [3 marks]

(iii) the rate, in metres per minute, at which the tide is falling 75 minutes after high tide. [5 marks]

Total 20 marks

02134020/CAPE 2007

GO ON TO THE NEXT PAGE
6. (a) (i) Use the result \( \int_{0}^{a} f(x) \, dx = \int_{0}^{a} f(a-x) \, dx, a > 0 \), to show that if \( I = \int_{0}^{\pi/2} \sin^2 x \, dx \), then \( I = \int_{0}^{\pi/2} \cos^2 x \, dx \).  

[ 2 marks] 

(ii) Hence, or otherwise, show that \( I = \frac{\pi}{4} \).  

[ 6 marks] 

(b) (i) Sketch the curve \( y = x^2 + 4 \).  

[ 4 marks] 

(ii) Calculate the volume created by rotating the plane figure bounded by \( x = 0, y = 4, y = 5 \) and the curve \( y = x^2 + 4 \) through \( 360^\circ \) about the \( y \)-axis.  

[ 8 marks] 

Total 20 marks

END OF TEST